LIQUID FLOW ON A VERTICAL PLANE WITH ABRUPT CHANGE IN BOUNDARY CONDITIONS

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We will consider the flow of a layer of viscous incompressible liquid along a vertical plane formed of two semiplanes abutting along a horizontal straight line. One semiplane moves relative to the other along their common boundary line. An exact analytic solution is obtained for the velocity distribution in the flow. The solution can be generalized to the case in which the plane consists of several close-spaced bands moving relative to each other along their boundary lines.

Without limiting generality, we will assume that the upper semiplane is at rest while the lower one moves. The dimensionless equations of motion and continuity and the boundary conditions have the form [1]

$$v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}\right), \tag{1}$$

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)+\frac{1}{\mathrm{Fr}},$$

$$v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right),\ \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0;$$

$$y=0\quad \begin{array}{l} x<0\ v_x=v_y=v_z=0,\\ x>0\ v_x=v_y=0,\ v_z=1; \end{array} \tag{2}$$

$$y=h(x)\quad \left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)(1-h_x^2)-4h_x\frac{\partial v_y}{\partial y}=0, \tag{3}$$

$$\frac{1}{\mathrm{Re}}\frac{\partial v_y}{\partial y}(1-h_x^2)-\frac{1}{\mathrm{Re}}h_x\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)-\frac{\alpha h_{xx}}{1-h_x^2}=p\,(1+h_x^2),$$

$$v_y=h_xv_x,\quad \partial v_z/\partial y=0;$$

$$x\to\mp\infty\quad v_x=\mathrm{Re}(-y^2/2+y)/\mathrm{Fr},\ v_y\to 0,\ v_z\to\delta, \tag{4}$$

$$\delta=\begin{cases}0, & x<0,\\ 1, & x>0,\end{cases}\quad \mathrm{Re}=U_0h_0/\nu,\quad \mathrm{Fr}=\frac{U_0^2}{gh_0},\quad \alpha=\frac{\sigma}{h_0\rho U_0^2},$$

where as length and velocity scales we use the thickness of the liquid layer h_0 as $x\to-\infty$ and the velocity of motion of the lower semiplane U_0 relative to the upper semiplane) z, y, and x are Cartesian coordinates (x is directed downward along the liquid flow, y is perpendicular to the plane, and z is along the boundary line between the planes, with origin on the boundary); v_x, v_y, and v_z are the velocity components along x, y, and z respectively; p is pressure; $h(x)$ is the liquid layer thickness; $h_x = dh/dx$; Re is the Reynolds number; Fr is the Froude number; σ is the surface tension coefficient; g is the acceleration of gravity; ρ is the liquid density; ν is the kinematic viscosity. The subscript zero denotes dimensional quantities.

The equations and boundary conditions are written with consideration of the fact that the flow is independent of the coordinate z. Boundary conditions (2) are conditions of adhesion to and non-flow through the solid surface, while boundary conditions (3) describe the equality of tangent and normal components of the stress tensors on the curved free surface h(x) of the liquid and gas phase.

We will seek a solution to the problem of Eqs. (1)–(4) in the form

$$v_x=\mathrm{Re}(-y^2/2+y)/\mathrm{Fr},\ v_y=0,\ v_z=v_z(x,y),\ h(x)=1. \tag{5}$$

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With substitution of Eq. (5), problem (1)–(4) reduces to solution of a single equation with boundary conditions:

$$\frac{\mathrm{Re}}{\mathrm{Fr}}\left(-\frac{y^2}{2}+y\right)\frac{\partial v_z}{\partial x}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}\right), \tag{6}$$

$$y=0 \quad v_z=\Theta(x), \qquad y=1 \quad \partial v_z/\partial y=0,$$

$$x\to-\infty \quad v_z\to 0, \qquad x\to\infty \quad v_z\to 1.$$

Here $\Theta(x)$ is a Heaviside function ($\Theta(x)=1,\ x>0;\ \Theta(x)=0,\ x<0$). Problems analogous to Eq. (6) arise in the theory of heat transfer [2, 3]. Performing the substitution $v_z = v(x, y) + \Theta(x)$ and using a Fourier transform, we reduce Eq. (6) to the form

$$\frac{\partial^2 V}{\partial y^2}+i\xi\left(i\xi+\frac{\mathrm{Re}^2}{\mathrm{Fr}}\left(-\frac{y^2}{2}+y\right)\right)V=i\xi+\frac{\mathrm{Re}}{\mathrm{Fr}}\left(-\frac{y^2}{2}+y\right), \tag{7}$$

$$y=0 \quad V=0, \ y=1 \quad \partial V/\partial y=0,$$

$$V(y,\xi)=\int_{-\infty}^{\infty} v(x,y)\,e^{i\xi x}dx.$$

The solution of Eq. (7) has the form

$$V(y,\xi)=F\left[\left(\frac{1}{16}\sqrt{\frac{i\xi D}{2}}-\frac{\xi^2}{4}+\frac{i\xi D}{8}\right)\Big/\sqrt{\frac{i\xi D}{8}},\ \frac{1}{8};\ -\sqrt{\frac{i\xi D}{2}}$$
$$\times(y-1)^2\right]F^{-1}\left[\left(\frac{1}{16}\sqrt{\frac{i\xi D}{2}}-\frac{\xi^2}{4}+\frac{i\xi D}{8}\right)\Big/\sqrt{\frac{i\xi D}{2}},\ \frac{1}{8};\ -\sqrt{\frac{i\xi D}{2}}\right]$$
$$\times\{1-\exp\{-[1+(y-1)^2]/2\}\}(i\xi)^{-1}, \quad D=\mathrm{Re}^2/\mathrm{Fr}. \tag{8}$$

Here F(a, c; z) is a Cummer function [4] and the square root should be considered as a single-valued function, coinciding on the upper side of the section along the real positive axis with the arithmetic value of the root. The distribution of the function v(x, y) is defined by the reverse Fourier transform of the function V(y, ξ)

$$v(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}V(y,\xi)\,d\xi. \tag{9}$$

From Eqs. (8) and (9) we find

$$v(x,y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-sx}}{s}\left\{1-\exp\left\{-\frac{1}{2}\left[(y-1)^2+1\right]\right\}\right.$$
$$\times F\left[\left(\frac{1}{16}\sqrt{\frac{sD}{2}}+\frac{s^2}{4}+\frac{sD}{8}\right)\Big/\sqrt{\frac{sD}{2}},\ \frac{1}{8};\ -\sqrt{\frac{sD}{2}}(y-1)^2\right]$$
$$\times F^{-1}\left.\left[\left(\frac{1}{16}\sqrt{\frac{sD}{2}}+\frac{s^2}{4}+\frac{sD}{8}\right)\Big/\sqrt{\frac{sD}{2}},\ \frac{1}{8};\ -\sqrt{\frac{sD}{2}}\right]\right\}ds, \quad s=i\xi. \tag{10}$$

From analysis of the behavior of the integrand of Eq. (10) we obtain formulas for the distribution $v_z(x, y)$ at x < 0 and x > 0.

For x < 0.

$$v_z(x,y)=\sum_{n=1}^{\infty}A_n e^{\omega_n x}Q_n(y), \tag{11}$$

$$Q_n(y)=\exp\left\{-\frac{1}{2}[(y-1)^2+1]\right\}F\left(\left[\frac{i}{16}\sqrt{\frac{\omega_n D}{2}}\right.\right.$$
$$\left.\left.+\frac{\omega_n^2}{4}-\frac{\omega_n D}{8}\right]\Big/i\sqrt{\frac{\omega_n D}{2}},\ \frac{1}{8};\ -i\sqrt{\frac{\omega_n D}{2}}(y-1)^2\right),$$

$$A_n=-\omega_n^{-1}[\partial H(\eta)/\partial\eta]^{-1}_{\eta=\omega_n},$$

$$H(\eta)=F\left(\left[\frac{i}{16}\sqrt{\frac{\eta D}{2}}+\frac{\eta^2}{4}-\frac{\eta D}{8}\right]\Big/i\sqrt{\frac{\eta D}{2}},\ \frac{1}{8};\ -i\sqrt{\frac{\eta D}{2}}\right).$$

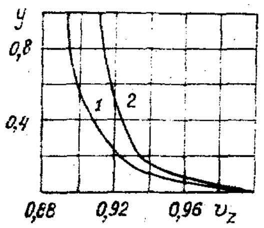

Fig. 1

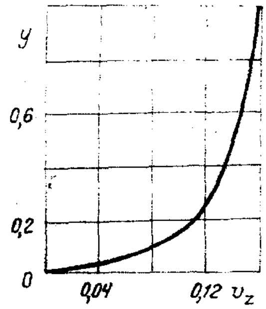

Fig. 2

Here ω_n are roots of the equation $H(\omega_n) = 0$, n = 1, 2, 3, . . ., lying in the right semiplane of the complex plane. We note that A_n can be represented in the form of the double series

$$A_n = \left\{\omega_n \sum_{n=0}^{\infty} \left[\frac{i}{32}\sqrt{\frac{D}{2\omega_n}} + \frac{\omega_n}{2} - \frac{D}{8} + \frac{in}{2}\sqrt{\frac{D}{2\omega_n}}\right] \times \left[\frac{i}{16}\sqrt{\frac{\omega_n D}{2}} + \frac{\omega_n^2}{4} - \frac{\omega_n D}{8} + in\sqrt{\frac{D\omega_n}{2}}\right]^{-1} \sum_{k=n+1}^{\infty} \left(\left[\frac{i}{16} \times \sqrt{\frac{\omega_n D}{2}} + \frac{\omega_n^2}{4} - \frac{\omega_n D}{8}\right] \Big/ i\sqrt{\frac{\omega_n D}{2}}\right)_k \left[-i\sqrt{\frac{\omega_n D}{2}}\right]^k \Big/ \left[\left(\frac{1}{8}\right)_k k!\right]\right\}^{-1}, \tag{12}$$

where $(a)_k$ is the Pochhammer symbol [4].

Similarly, from Eq. (10) with consideration of $v_z = v(x, y) + \Theta(x)$ we obtain the velocity distribution for $x > 0$

$$v_z(x, y) = \sum_{n=1}^{\infty} B_n e^{-\sigma_n x} G_n(y) + 1,$$

$$G_n(y) = \exp\left\{-\frac{1}{2}[(y-1)^2 + 1]\right\} F\left[\left(\frac{1}{16}\sqrt{\frac{sD}{2}} + \frac{s^2}{4} + \frac{sD}{8}\right) \times \left(\frac{sD}{2}\right)^{-1/2}, \frac{1}{8}; -\sqrt{\frac{-sD}{2}}(y-1)^2\right], \quad B_n = \sigma_n^{-1}\left[\frac{dR(\eta)}{d\eta}\right]^{-1}_{\eta=\sigma_n},$$

$$R(\eta) = F\left[\left(\frac{1}{16}\sqrt{\frac{\eta D}{2}} + \frac{\eta^2}{4} + \frac{\eta D}{8}\right)\Big/\sqrt{\frac{\eta D}{2}}, \frac{1}{8}; -\sqrt{\frac{\eta D}{2}}\right], \tag{13}$$

where σ_n is the n-th positive root of the transcendental equation $R(\eta) = 0$. The representation of B_n in the form of a double series is

$$B_n = \left\{\sigma_n \sum_{n=0}^{\infty}\left(\frac{1}{32}\sqrt{\frac{D}{2\sigma_n}} + \frac{\sigma_n}{2} + \frac{D}{8} + n\sqrt{\frac{D}{2\sigma_n}}\right)\left(\frac{1}{8}\sqrt{\frac{\sigma_n D}{2}} + \frac{\sigma_n^2}{4} + \frac{\sigma_n D}{8} + n\sqrt{\frac{\sigma_n D}{2}}\right)^{-1} \sum_{k=n+1}^{\infty}\left[\left(\frac{1}{16} \times \sqrt{\frac{\sigma_n D}{2}} + \frac{\sigma_n^2}{4} + \frac{\sigma_n D}{8}\right)\Big/\sqrt{\frac{\sigma_n D}{2}}\right]_k \left(-\sqrt{\frac{\sigma_n D}{2}}\right)^k \Big/ \left[\left(\frac{1}{8}\right)_k k!\right]\right\}^{-1}. \tag{14}$$

Thus, Eqs. (5) and (11)–(14) define the analytical solution of the problem of distribution of the velocity of a liquid flowing down a vertical plane with abrupt change in boundary conditions. It is evident from Eqs. (11)–(14) that the distribution $v_z(x, y)$ depends solely on the combination of the Reynolds and Froude numbers: $D = Re^2/Fr$.

We note that a similar solution can be obtained for the case of any constant inclination of the planes over which the liquid layer moves. Moreover, if in Eq. (6) we replace the velocity v_z with the temperature T, and the ratio Re^2/Fr by the Peclet number Pe, then the problem reduces to determination of the temperature profile in a liquid moving between two planes on which $T = T_1$ for $x < 0$ and $T = T_2$ at $x > 0$. In this case, Eqs. (11)–(14) provide a solution for the temperature. The corresponding temperature profile problem was solved approximately in [3] for small Peclet numbers by using expansion in a series.

Equations (11)–(14) were used to calculate the distribution $v_z(x, y)$ for values of the parameter D = 1. Initially roots of the functions $H(\eta)$ and $R(\eta)$ were found, then roots of the functions $R(\eta)$ were determined by Mueller's iterative method for solution of algebraic equations of high degree [5]. After finding the roots of the functions $H(\eta)$ and $R(\eta)$ Eqs. (11)–(14) were summed numerically and their real components were determined. Figure 1 shows the distribution $v_z(x, y)$ along the y-axis for D = 1 and x = 0.4 (curve 1), x = 0.8 (curve 2).

As is evident from Fig. 1, the $v_z(x, y)$ curves decrease monotonically with increase in y. The upper liquid layers on the segment adjacent to the boundary lag behind the lower. But with increase in x the distribution $v_z(x, y)$ tends to unity. The velocity in the z direction over the entire liquid thickness becomes equal to the velocity of the moving semiplane.

Figure 2 shows $v_z(x, y)$ along the y-axis for D = 1 and x = −0.01. According to Fig. 2, the curve $v_z(x, y)$ monotonically increases from 0 at the point y = 0 to 0.1577 at y = 1.

Viscous stresses acting in the z-direction and created by the liquid flow in the lower semiplane region lead to a motion along z of the upper liquid layers over the fixed semiplane. The liquid flow over two semiplanes moving relative to each other occurs without change in the liquid layer thickness or the velocity component distribution along the x-axis.

In conclusion, we will consider liquid flow on a plane consisting of several adjoining regions: $x < x_1$, $x_1 \leqslant x \leqslant x_2$, . . ., $x_i \leqslant x \leqslant x_{i+1}$, $x > x_{i+2}$, moving relative to each other with velocities U_k (k = 1, . . ., i +2).

For simplicity, without limiting generality we will consider the case in which the upper region x < 0 is at rest, the adjacent infinite band $0 \leqslant x \leqslant x_0$ moves along the boundary line with a velocity ϑ_1, and the lower region $x > x_0$ moves with a velocity ϑ_2 relative to the fixed region. The flow along the x-axis does not change, and the flow along the z-axis can be found by solution of the problem

$$\frac{\mathrm{Re}}{\mathrm{Fr}}\left(-\frac{y^2}{2}+y\right)\frac{\partial v_z}{\partial x}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}\right); \tag{15}$$

$$y=0 \quad v_z=\begin{cases}0, & x<0,\\ \vartheta_1, & 0<x<x_0,\\ \vartheta_2, & x>x_0,\end{cases} \quad y=1 \quad \frac{\partial v_z}{\partial y}=0; \tag{16}$$

$$x\to-\infty \quad v_z\to 0, \quad x\to\infty \quad v_z\to\vartheta_2. \tag{17}$$

Since Eq. (15) and boundary conditions (16) and (17) are linear, the solution of Eqs. (15)–(17) can be sought in the form of the sum of solutions of two problems: the solution of the first problem describes the flow on a plane consisting of two adjacent semiplanes; the upper at rest, and the lower moving with a velocity of ϑ_1, while the solution of the second problem describes a flow in which the upper region $x < x_0$ is at rest, and the lower moves with a velocity ϑ_3, such that $\vartheta_1+\vartheta_3=\vartheta_2$. Thus, the solution of Eqs. (15)–(17) reduces to solution of the following problems:

Eq. (15) with boundary conditions

$$y=0 \quad v_z=\begin{cases}0, & x<0,\\ \vartheta_1, & x>0,\end{cases} \quad y=1 \quad \frac{\partial v_z}{\partial y}=0,$$

$$x\to-\infty \quad v_z\to 0, \quad x\to\infty \quad v_z\to\vartheta_1;$$

Eq. (15) with boundary conditions

$$y=0 \quad v_z=\begin{cases}0, & x<x_0,\\ \vartheta_3, & x>x_0,\end{cases} \quad y=1 \quad \frac{\partial v_z}{\partial y}=0,$$

$$x\to-\infty \quad v_z=0, \quad x\to\infty \quad v_z\to\vartheta_3, \quad \vartheta_3=\vartheta_2-\vartheta_1.$$

The method described can be generalized to the case of several regions moving relative to each other.

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BREAKUP OF A FREE JET OF A VISCOELASTIC FLUID

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A large number of papers has been published on the breakup of a jet of a high-viscosity Newtonian liquid flowing in a low-viscosity medium. The results of these papers show that there exist three regimes of jet breakup, depending on the flow velocities. At low velocities it breaks up under the action of capillary forces, while the long-wave axially symmetric perturbations increase most quickly.

The nature of perturbation evolution changes with increasing velocity. When the dynamic action of the medium exceeds the capillary forces the long-wave bending perturbations increase significantly more quickly than the axially symmetric ones. With further velocity increase the jet breakup into large parts is changed by spraying into a set of small droplets, the size of which is independent of the jet radius. The main purpose of the present work is to determine the velocity range in which a jet of a viscoelastic liquid breaks up into large parts.

We investigate the evolution of long-wave perturbations $kR \leqslant 1$ in a circular jet of radius R of a viscoelastic liquid of density ρ, flowing with velocity U in a low-viscosity medium of density ρ_0, where k is the perturbation wave number. If for $kR \leqslant 1$ the perturbation increment increases monotonically with kR, the nature of the breakup is approximated by a spray, requiring the study of short-wave asymptotics. The analysis is based on the equations derived in [1]. We consider jets undergoing primarily extension or compression. This problem was first formulated in [2]. This mathematically insignificant complication of the problem makes it possible to estimate qualitatively the effect of a longitudinal strain, occurring in a viscoelastic jet, on its stability. We choose the rheological equation of a Maxwell liquid with viscosity η and relaxation time λ [3]: $\mathbf{T} + \lambda(D\mathbf{T}/Dt - \mathbf{W}\cdot\mathbf{T} + \mathbf{T}\cdot\mathbf{W}) + \lambda\varepsilon\cdot(\mathbf{D}\cdot\mathbf{T} + \mathbf{T}\cdot\mathbf{D}) = 2\eta\mathbf{D}$, where D/Dt is the convective derivative, $\mathbf{D}$ is the velocity deformation tensor, $\mathbf{W}$ is the vorticity tensor, $\mathbf{T}$ is the stress tensor, and ε = 0, 1, and −1, respectively, for the Jaumann, lower, and upper convective derivatives. The Maxwell liquid model is the simplest model qualitatively describing many properties of polymer liquids: instantaneous elasticity, stress relaxation, difference in normal stresses, etc. [3]. The equations for the additional capillary and hydrodynamic pressures occurring during perturbation of a jet surface $r = R + \zeta(\varphi, z, t)$ are taken from [4-6]. The system of equations for small perturbations is significantly simplified when it is not necessary to take into account the time dependence of the longitudinal stress in the jet. This assumption is valid for $t/\lambda \ll 1$. In the absence of longitudinal stress these equations describe the evolution of perturbations in a relaxing jet. A solution of the equations is sought in the form $\exp(ikz + \alpha t)$. The system of equations decomposes into separate systems, each of which corresponds to a definite perturbation. We provide the equations for the azimuthal dependence of the surface displacement and the corresponding dispersion equations. The perturbations retaining the jet linearity are:

$$\zeta = \zeta_0 e^{\pm i2\varphi}, \frac{1}{4}\rho\alpha^2 R^2 + \frac{\alpha}{1+\alpha\lambda}\left(2\eta_1 + \eta_2\frac{k^2R^2}{4}\right) - \frac{R^2}{2}V(k, 2) = 0,$$

$$\zeta = \zeta_0 e^{\pm i3\varphi}, \frac{\rho\alpha^2 R^2}{24} + \frac{\alpha}{1+\alpha\lambda}\left(\eta_1 + \eta_2\frac{k^2R^2}{24}\right) - \frac{R^2}{8}V(k, 3) = 0,$$

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